

Divergent modes of online collective attention to the COVID-19 pandemic are associated with future caseload variance

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Using a random 10% sample of tweets authored from 2019-09-01 through 2020-03-25, we analyze the dynamic behavior of words (1-grams) used on Twitter to describe the ongoing COVID-19 pandemic. Across 24 languages, we find two distinct dynamic regimes: One characterizing the rise and subsequent collapse in collective attention to the initial Coronavirus outbreak in late January, and a second that represents March COVID-19-related discourse. Aggregating countries by dominant language use, we find that volatility in the first dynamic regime is associated with future volatility in new cases of COVID-19 roughly three weeks (average 22.7 ± 2.17 days) later. Our results suggest that surveillance of change in usage of epidemiology-related words on social media may be useful in forecasting later change in disease case numbers, but we emphasize that our current findings are not causal or necessarily predictive.

I. INTRODUCTION

COVID-19 is a potentially lethal viral respiratory disease that is causing a global pandemic [1, 2]. While Coronavirus testing availability remains deeply problematic, social media data can be part of an effective strategy for infectious disease surveillance [3–8]. Previous work has demonstrated that online collective attention to COVID-19 as measured by social media activity has fluctuated from the date of the first public report of the disease (2019-12-31) to near the time of writing (2020-03-25) [9–11].

In this work we analyze time series of word (1-gram) ranks on Twitter computed from a 10% random sample of all messages. We find that the temporal dynamics of this discourse separate into two distinct clusters, one (C_1) that contains words contributing to the explosive rise in online discussion of COVID-19 prevention and treatment during March 2020 and another (C_2) that contains words contributing to the rise and subsequent fall in collective attention to COVID-19 during mid-January – mid-February 2020. Variance of percent changes in word time series closest to the centroid of C_2 is a consistent leading indicator of variance in percent change in new cases of COVID-19. We close with a short discussion of the implications and limitations of these findings, and suggestions for future research [12].

II. DATA

We analyzed time series of word usage on a random 10% sample of tweets written between 2019-09-01 and

2020-03-25. For each language under study, we considered only the top 1000 words used in the language as ranked during the first three weeks of March 2020 [11], and restrict our analysis to the same 24 languages analyzed in a previous work. Languages are detected and annotated using a previously-introduced procedure [13]. We obtained data on languages spoken in each country from the Australian Federal Department of Social Services and data on number of new COVID-19 cases by country from the European Centers for Disease Control [14].

III. RESULTS

A. Divergent modes of COVID-19 related language

We find $k^* = 6$ clusters of normalized log rank word usage timeseries using the algorithm detailed in Sec. V A. We compute these clusters using the entire dataset, i.e., aggregating all log rank time series in each of the 24 languages under study. Of these clusters, four are composed primarily of words that do not appear to relate to COVID-19. Though the centroid of one of these clusters correlates with the rise of COVID-19 related language, manual inspection of word time series close in Euclidean distance to that cluster centroid are related to political events (e.g., the United States Democratic Primary) that occurred simultaneously.

The remaining two clusters contain language that relates to COVID-19 both explicitly and implicitly. We label these clusters C_1 and C_2 and their cluster centroids $E[C_1]$ and $E[C_2]$ respectively. (The ordering of the cluster subscripts comes from the respective maxima of their cluster centroids.) $E[C_1]$ exhibits very little variation until the first week of March 2020, where it begins a

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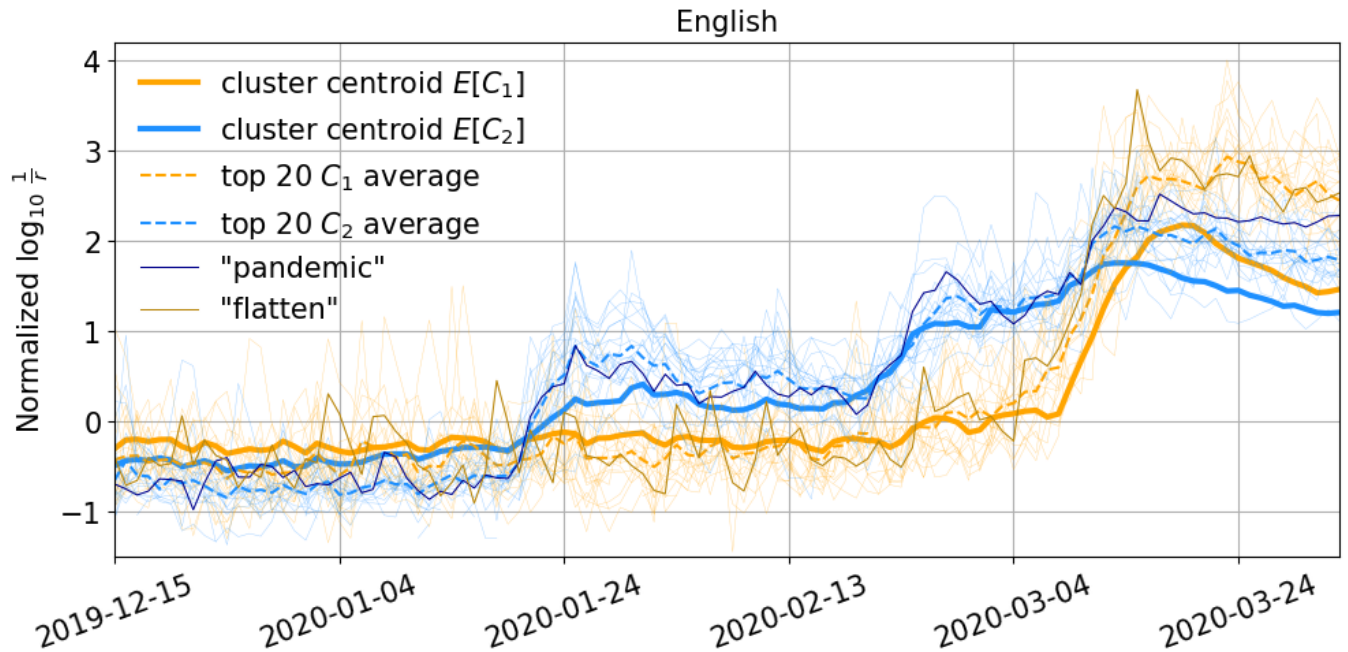


FIG. 1. The rise in collective attention to COVID-19 during late January 2020 to early February 2020 followed by a marked decline preceding the global pandemic is generated by two distinct clusters of COVID-19 related language. We display COVID-19 related cluster centroids in thick curves, $E[C_1]$ in a lighter hue and $E[C_2]$ in a darker hue, the mean normalized log rank timeseries of the top 20 English words closest to each of $E[C_1]$ and $E[C_2]$ in thinner, dashed curves, and the single English word closest to each of $E[C_1]$ and $E[C_2]$ in thin solid curves. Increasing granularity (centroids \rightarrow top 20 words \rightarrow single representative word) is associated with exaggeration of the dynamics of $E[C_1]$ and $E[C_2]$. Before normalization we map $\log_{10} r \mapsto \log_{10} \frac{1}{r}$ so that higher values on the vertical axis are lower values of (log) rank. For comparison, the word “pandemic” rises in popularity from a rank of 133,445 on December 21 to a rank of 188 on March 17, while the word “flatten” goes from being the 100,913th most popular English word on January 20 to being 2,131st most used word on March 15.

sustained increase in time. Conversely, $E[C_2]$ exhibits a smaller increase at the end of January 2020 followed by a larger increase in the second week of February 2020. This second increase in $E[C_2]$ is followed by another sustained increase until mid March 2020.

This divergent dynamic behavior is amplified when restricting analysis to sets of individual word time series that are closest to $E[C_1]$ or $E[C_2]$ in Euclidean distance. The mean normalized log rank timeseries of the top 20 words in each language that were closest to $E[C_1]$ and $E[C_2]$ exhibit the same qualitative behavior for most of the 24 languages under study, but this behavior is amplified (greater magnitudes of increase and decrease). We display these dynamics for English in Fig. 1 and for all 24 languages under study in Figs. 2 and 3. We plot languages in order of frequency of usage on Twitter in Figs. 2, 3, 4, and 5. For interpretable visualization, we invert ranks ($\log_{10} r \mapsto \log_{10} \frac{1}{r}$) before normalization and before plotting, so that lower ranked words — words that are more popular and are receiving more attention — are higher on the vertical axis than words of higher rank corresponding to lower popularity. We display the top 20 words associated with C_1 and C_2 in Tab. I.

Words assigned to C_1 reflect immediate measures taken

to prevent the spread of COVID-19, such as “flatten”, “distancing”, “tltravail” (telework), “hospitalier” (hospital), “encerrado” (closed), and “evitar” (to avoid).

In contrast, words assigned to C_2 include more conceptual words that describe people, agencies, institutions, and concepts surrounding epidemics more generally, such as “pandemic”, “CDC”, “epidemiologist”, “l’pidmie” (the epidemic), “virus”, “contagiado” (contagious). Words assigned to C_2 describe pandemics in general, while words assigned to C_1 describe quarantines and lockdowns in particular. Though we have not conducted a formal linguistic analysis to conclude that there are significant semantic differences between words assigned to each cluster, these preliminary findings provide evidence that such a semantic difference does exist.

B. Death attribution by language

Using the methodology described in Sec. VB, we aggregate country-wide infection case numbers and bin them approximately by language, thus enabling analysis of new cases stratified by language. Because the log word rank time series are nonstationary and the new case number time series are not scale-independent, we move

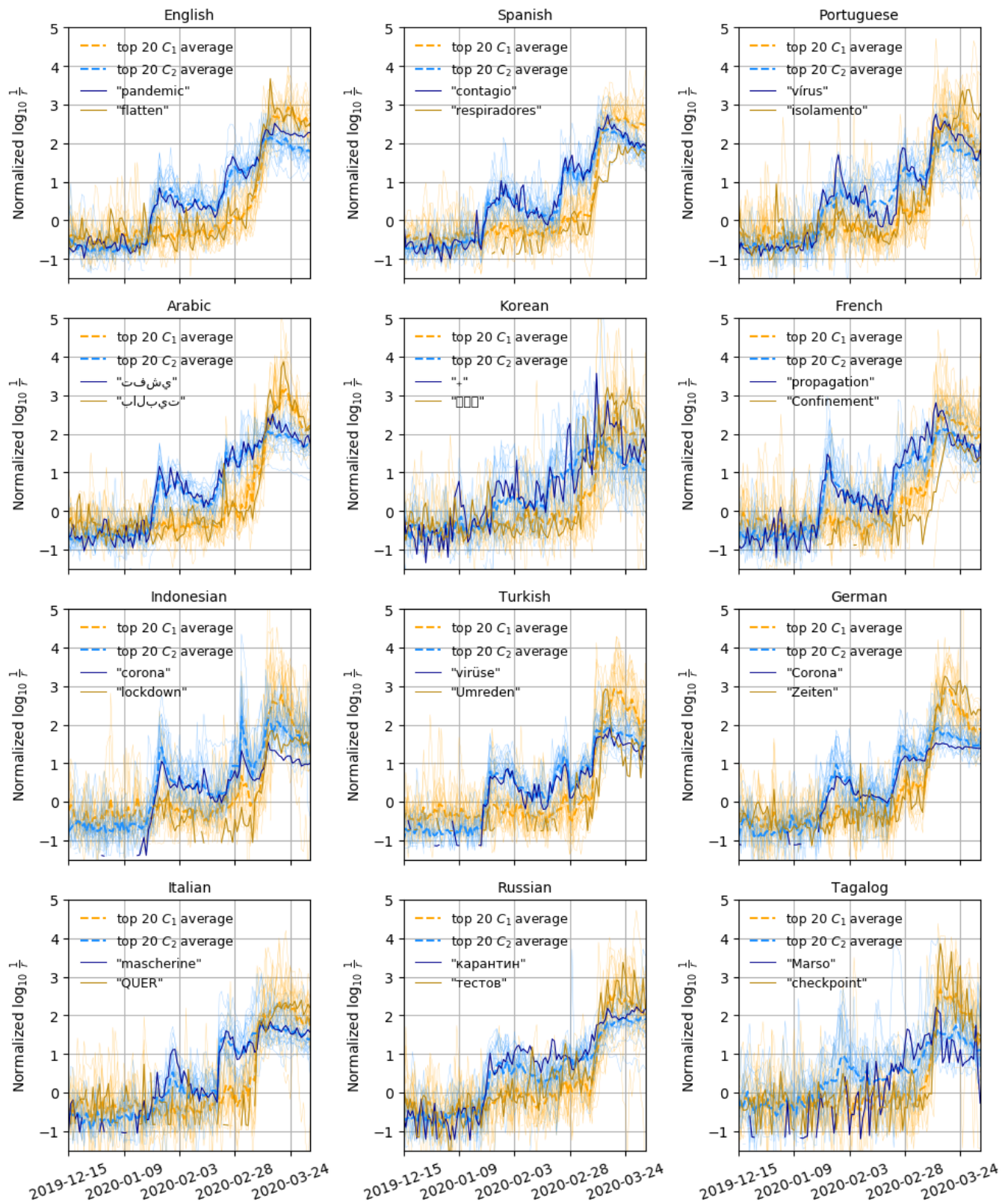


FIG. 2. We display the mean normalized log rank timeseries of the top 20 words closest to each of $E[C_1]$ and $E[C_2]$ in dashed curves and the single word closest to each of $E[C_1]$ and $E[C_2]$ in thin solid curves for each of the first 12 of 24 languages. The divergent modes of dynamic behavior are consistent across most languages, with some languages (English, French, German, and Indonesian) displaying prominently larger peaks in words closest to $E[C_2]$ during late January through early February 2020. Other languages, such as Korean and Tagalog, do not display this behavior.

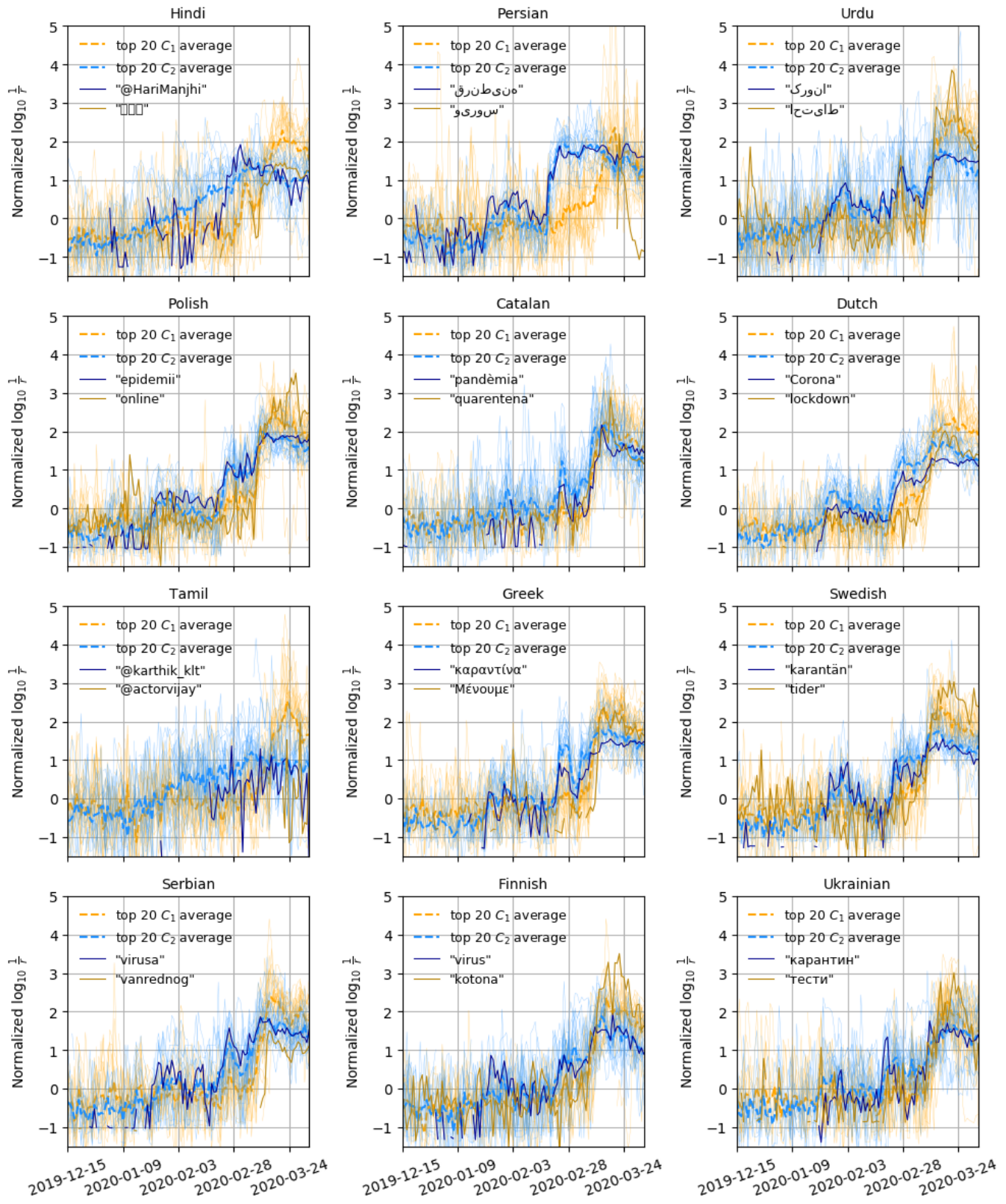


FIG. 3. For the second 12 of 24 languages, we display the mean normalized log rank timeseries of the top 20 words closest to each of $E[C_1]$ and $E[C_2]$ in dashed curves and the single word closest to each of $E[C_1]$ and $E[C_2]$ in thin solid curves. (We display the first 12 of 24 languages in Fig. 2.)

Top 20 closest to $E[C_1]$	Top 20 closest to $E[C_2]$
flatten	pandemic
home	cases
positive	Pandemic
distancing	virus
isolation	corona
Distancing	Corona
virtual	Virus
essential	pandemics
ventilator	quarantine
curve	Fauci
ventilators	epidemiologist
workers	CORONA
stay	infected
Lockdown	infections
lockdown	outbreaks
Tested	quarantined
crisis	CDC
businesses	disinfecting
Bergamo	#Corona
flattening	@WHO

TABLE I. We display the top 20 English words closest to the centroids of C_1 and C_2 . The pattern of words assigned to C_1 being more specific and concerned with social distancing in particular, while words assigned to C_2 are focused on pandemics more generally, is apparent. Though individual words may change rank, this list is qualitatively insensitive to regeneration of clusters as new data becomes available.

to a percent-change based analysis of these data. We analyze percent-change time series for the log word rank closest to $E[C_2]$ and the new case time series for each language. We estimate a latent volatility statistic for each percent-change time series. This statistic captures the latent variance of the time series at each point in time without destroying information through computation of a rolling variance.

Peaks in the latent volatility statistic indicate days on which the underlying time series exhibited large percent-changes in its value. We measure the distance between (a) the peak latent volatility statistic of percent-change log word rank and (b) the peak latent volatility statistic of percent-change new cases, termed the peak-to-peak distance (P2PD), for each language under study. P2PD is a simple metric of the lag between fluctuations in social media attention to the initial Coronavirus outbreak and (usually positive) large fluctuations in new cases. Mean P2PD is approximately 22.7 days with a estimated SEM of 2.17 days. Moreover, P2PD is greater than zero for all but two languages (Tamil and Portuguese) under study.

Observed values of P2PD are statistically reproduced by a simple Poisson data generating process. We model the days at which peak volatility of each percent-change time series occurred as being generated by a Poisson distribution with an unknown rate parameter. P2PD is then modeled as the number of days between the peak day of new case volatility and the peak day of log word rank

volatility. The observed cumulative distribution function (cdf) of P2PD data is contained within the posterior distribution of this model’s empirical cdfs. We display the distribution of estimated mean P2PD in Fig. 6 and empirical cdfs generated by the posterior predictive distribution of the difference-of-Poissons model in Fig. 7.

IV. DISCUSSION

Analyzing the behavior of words found in a random 10% sample of all tweets between 2019-09-01 and 2020-03-25, we find a distinct bilateral split in dynamics of words relating to the COVID-19 pandemic. Though we have not performed a formal linguistic analysis, evidence suggests that words used to describe the initial reports of a Coronavirus outbreak in China differ semantically from words used later to describe the worldwide fight against the pandemic. This second cluster reflects discussion of specific measures, such as quarantine and social distancing, currently being used to mitigate the spread of the virus and limit casualties.

The initial spike in collective attention to the Coronavirus in mid-January 2020, subsequently followed by a decay, is explained by the dynamics of the first cluster of words and not the second. The mean number of days between peak volatility of percent change in first-cluster words and peak volatility of percent change in new case numbers (P2PD) is approximately 23 days, which is comparable to estimates of right-censored median time delay between onset of COVID-19 and death [15, 16] and median duration of viral shedding [17]. The observed distribution of P2PD is statistically reproduced by a simple difference-of-Poissons model when aggregating across all languages under study.

This study is exploratory. We take care to not extrapolate from the current set of results without adequate caution. First, we use only the top 1000 words in each language as ranked in March 2020 when compared with March 2019 [11]. This list of words is dynamic and may change our results either quantitatively or qualitatively. Second, all of our results are non-causal because we analyze the entirety of each time series (word time series and new infection case numbers).

Associations that we find, such as a well-behaved Skellam distribution of P2PD, should not be taken as causal, or even necessarily predictive, for two reasons. It is obvious that change in word usage rank on Twitter does not cause new cases of COVID-19. Though it may be possible to use change in word usage rank to inform predictions of new case numbers, we have not performed such forecasting ourselves and it is possible that these results will not hold in the future. In addition, this time delay may be applicable only to COVID-19 and not necessarily other infectious diseases. While we have attempted to control for nonstationarity and explicit time dependence by analyzing percent changes and their variance — and not analyzing correlation between the nonstationary time

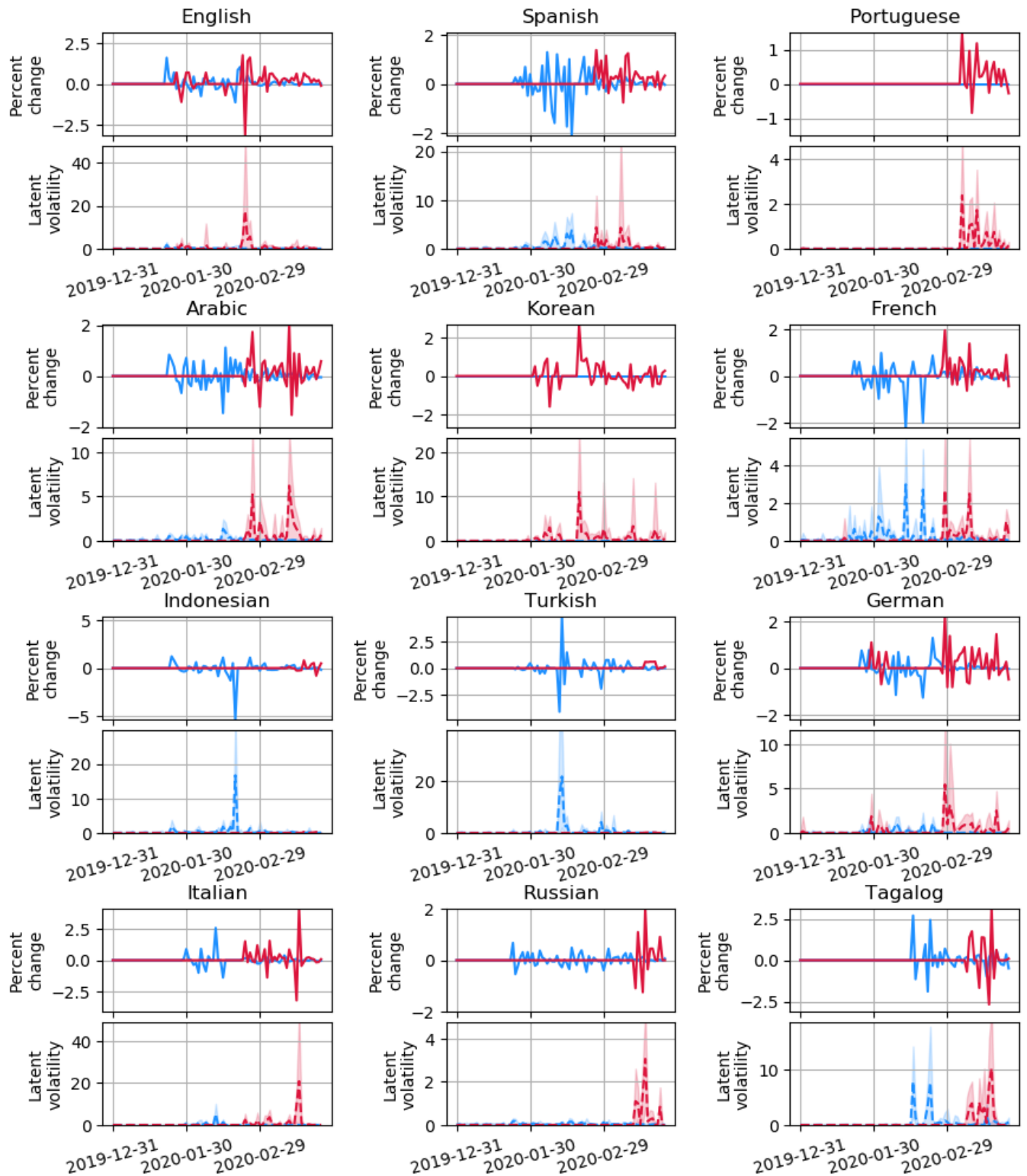


FIG. 4. We display percent-change time series and associated latent variance (volatility) time series for both log rank usage of the word closest to $E[C_2]$ (blue curves) and new case number time series (red time series) for each language. This figure presents the first 12 of 24 languages. There is a positive association between peak volatility in log rank word usage and future peak volatility in new infection case numbers. For all languages but two (Tamil and Portuguese), the peak-to-peak difference (P2PD) between case and log rank volatility is positive. The empirical cumulative distribution function (ecdf) of P2PD is reproduced by a simple difference-of-Poissons (Skellam) model, though the ecdf is mildly overdispersed compared to the mean ecdf of the Skellam model. The Skellam model does not reproduce the first three order statistics (Finnish, Russian, and Ukrainian).

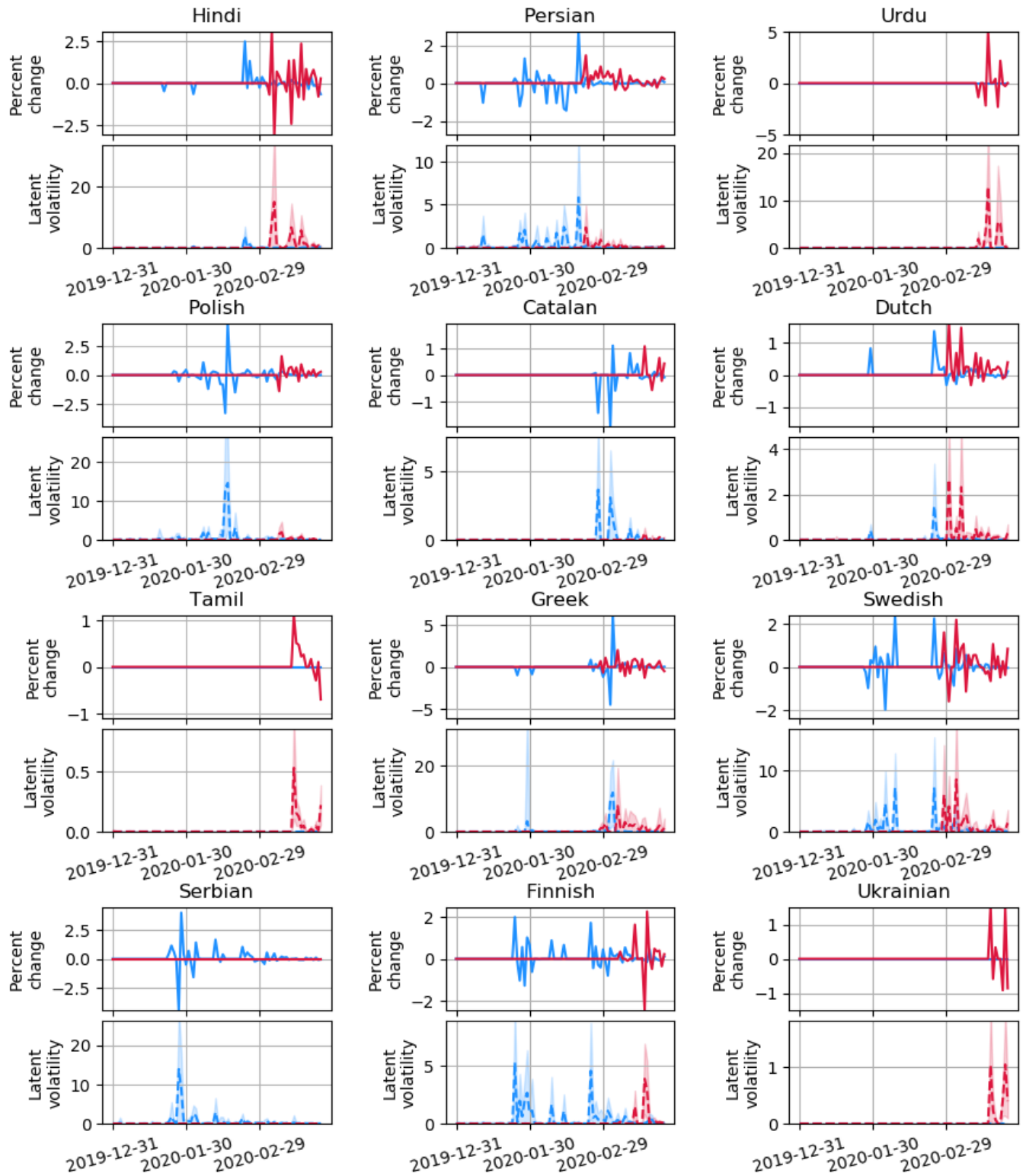


FIG. 5. The second 12 of 24 percent-change and latent volatility time series for log rank (blue time series) and new case load (red time series); we display the first 12 of 24 and provide an expanded description in Fig. 4.

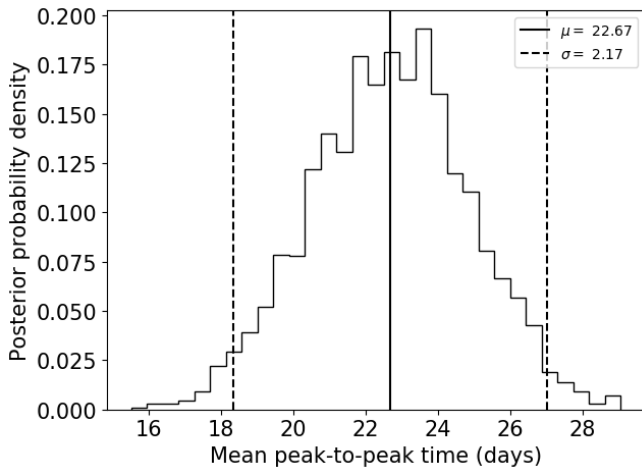


FIG. 6. Mean distance between the peak latent volatility of percent-change in new cases and peak latent volatility of percent-change of closest C_2 word to $E[C_2]$ is approximately $\mu = 22.7$ days, with a bootstrapped standard deviation of the mean given by $\sigma \approx 2.17$ days.

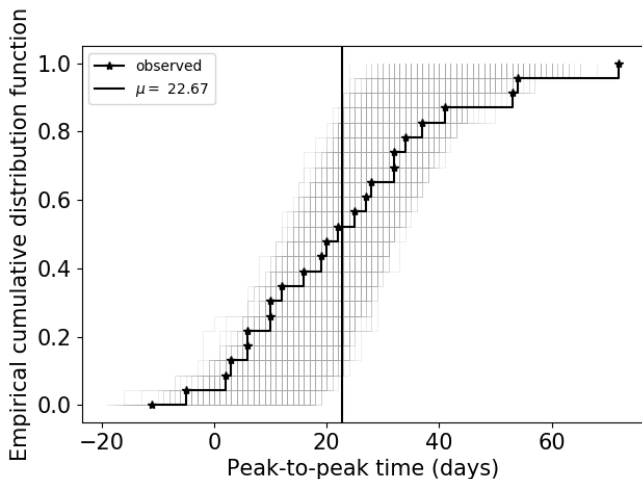


FIG. 7. The observed empirical cdf of the P2PD data is reproduced by the posterior predictive distribution of empirical cdfs of the Poisson model described in Sec. V B. The posterior mean μ of the P2PD data is displayed as a vertical black line. The data is overdispersed but is largely captured by the posterior distribution of empirical cdfs except for the languages with the three largest P2PDs (Finnish, Russian, and Ukrainian).

series of log word rank and new case numbers — this does not mean that the association is not spurious and more extensive analysis of this association is warranted.

While it is suggestive that mean P2PD is comparable to estimates of time delay between COVID-19 onset and death, we particularly hesitate to draw any conclusions from this observation, though it should be a target of further theoretical and empirical study. We do not have subject-matter expertise in epidemiology and so will not

offer speculation on this matter.

There are several ways in which this study could be extended, for example by continuously updating words through time in order to test our methods' generalizability. More importantly, the methodology can be applied to other infectious disease outbreak data to test our hypothesis that changes in social media attention to epidemic-related words can provide a useful signal in predicting future new case volatility. Future studies could also use more sophisticated clustering, similarity search or latent volatility estimation methods [18, 19].

V. METHODS

A. Cluster number selection

We clustered the log word rank time series $\log_{10} \frac{1}{r_t}$ using the minibatch k -means clustering (KMC) algorithm [6]. Before clustering, we normalized the time series so that clusters would not form purely based on the average rank of each word. The functional form of the normalization was $\log_{10} \frac{1}{r_t} \mapsto \frac{\log_{10} \frac{1}{r_t} - \mu}{\sigma}$, where $\mu = \frac{1}{T} \sum_{t=1}^T \log_{10} \frac{1}{r_t}$ and $\sigma^2 = \frac{1}{T} \sum_{t=1}^T (\log_{10} \frac{1}{r_t} - \mu)^2$.

We chose the number of clusters k^* using the following algorithm [20]. For each of N independent trials, we fit a minibatch KMC model for each of $k = 1, \dots, 15$ clusters. For each of these clusters in each independent trial, we recorded the average Euclidean distance of the set of all time series from the closest cluster centroid. We denote this error metric by $\ell_{n,k}$. We then computed a ratio-of-ratios statistic, $a_{n,k} = \frac{\ell_{n,k+1}}{\ell_{n,k}} / \frac{\ell_{n,k}}{\ell_{n,k-1}}$, and selected the number of clusters as $k^* = \arg \min \{k - 1 : a_k \leq 1\}$, where we have put $a_k = \frac{1}{N} \sum_{n=1}^N a_{n,k}$. We display bootstrapped single standard deviation confidence intervals around a_k in Fig. 8. This algorithm returned the number of clusters $k^* = 6$.

We display the centroids of the inferred clusters in Fig. 9. We used tweets authored both before and during the COVID-19 pandemic to generate the clusters, so the centroids are relatively flat before the initial coronavirus reports (late December 2019) and some exhibit periodic behavior. The magnitude of the horizontal axis is lower than in Figs. 1, 2, and 3 because here we display only the cluster centroids, which necessarily have moderated fluctuations compared to the more extreme cluster elements displayed in other figures.

B. Case number attribution by language

To associate country-level case number changes with languages, we performed a one-to-one lossy mapping of country to dominant language spoken in that country. Using data from the Australian federal government's Department of Social Services, we truncated the list of

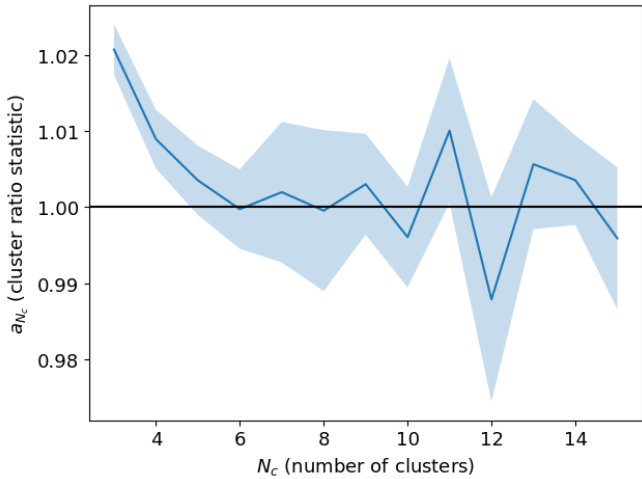


FIG. 8. Mean and bootstrapped confidence intervals for ratio statistic used in choosing number of clusters in minibatch k -means algorithm.

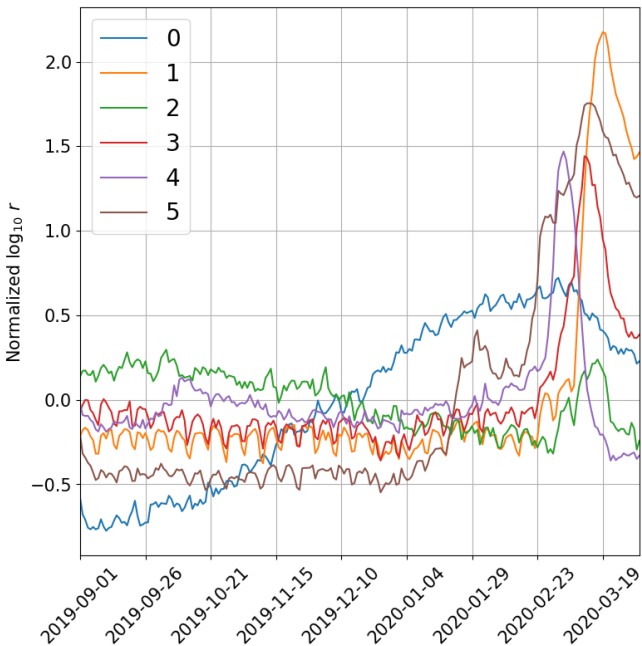


FIG. 9. Using the algorithm detailed in Sec. V A, we find $k^* = 6$ clusters of log rank word time series during the time period 2019-09-01 to 2020-03-25. We do not label the clusters with informative labels because clusters are unique only up to a permutation of labels.

languages spoken in each country to the most prevalent language in each country. While this mapping is crude and eliminates subtleties of intranational language diversity (e.g., Switzerland is mapped solely to German, while French, Italian, and Romansh are dropped), it allowed us to reverse the direction of this mapping and assign to

each language the number of new cases equal to the sum of new cases in each country for which the language is the primary language. We obtained new case numbers from the European Center for Disease Control and Prevention.

We move to a percent-change approach in our joint analysis of new case numbers and log rank word time series because log rank word time series are nonstationary (they are only wide-sense stationary in our analysis because we normalize them to have intertemporal zero mean and unit variance) and new case number time series are not scale-independent. We define the percent-change time series as $y_t \equiv \log \frac{x_t}{x_{t-1}}$, where $x_t \in \{\text{log rank word time series, new case time series}\}$. Instead of analyzing y_t , an unbounded random variable, we instead analyze the variance of y_t , denoted s_t , by estimating a standard Bayesian stochastic volatility model [21, 22]. We hypothesize that the latent log-variance s_t evolves according to

$$s_t \sim \text{Normal}(s_{t-1}, v^2), \quad s_0 \sim \text{Normal}(0, 1). \quad (1)$$

We place a weakly informative prior on the standard deviation of the increments of this process, $v \sim \text{LogNormal}(0, 1)$. The percent change is then modeled as

$$y_t \sim \text{Normal}(\mu, \exp(s_t/2)), \quad (2)$$

where we include a tight zero-centered prior for the mean percent change, $\mu \sim \text{Normal}(0, 0.01)$. We fit this model using stochastic variational inference with a diagonal normal guide (variational posterior) [23]. We conduct our analysis using draws from the trained guide distribution.

In addition to estimating the mean P2PD, we assess the likelihood that P2PD can be explained by a simple statistical model. We model number of days to peak in average latent volatility in each of the percent change time series as $N_s \sim \text{Poisson}(\lambda)$, where we place a weakly informative prior on the rate parameter, $\lambda \sim \text{LogNormal}(0, 1)$. We sample from this Poisson model for each of the percent change time series, and then model P2PD as the difference in these Poisson rvs (known as a Skellam distribution). We use the No U-Turn Sampler algorithm to sample from the posterior [24]. We display draws from the posterior predictive distribution of empirical cdfs in Fig. 7, along with the observed empirical cdf of the P2PD data.

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